ON CERTAIN ALGEBRAIC CYCLES ON ABELIAN VARIETIES OF FERMAT TYPE

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1. Introduction

Let X_m^n be the n-dimensional Fermat variety of degree m, which is defined by

$$X_m^n = \{(x_0 : \cdots : x_{n+1}) \in \mathbb{P}^{n+1} : x_0^m + \cdots + x_{n+1}^m = 0\}.$$

These objects have long been studied in algebraic geometry and number theory. In the late 1970s, T. Shioda developed arithmetic methods to study homological properties of algebraic cycles on Fermat varieties and other special varieties originated from Fermat varieties. For example, the Hodge conjecture for X_m^n was well-studied in [7] by using the inductive structures of Fermat varieties and the character decomposition of algebraic cohomology classes with respect to the finite abelian group G_m^n acting naturally on X_m^n. In [8], he continuously studied the Hodge conjecture for abelian varieties of Fermat type, which is isogenous to a product of certain factors of the Jacobian variety $J(X_m^1)$, and further studied homological properties of algebraic cycles on them via their canonical decompositions derived from the character decomposition of $H^1(X_m^1)$.

Let us introduce certain special algebraic cycles on certain products of Fermat curves and on their Jacobian varieties, which are constructed as follows. Let $\mathcal{A}^n = (H_{ij})_{0 \leq i < j \leq n+1}$ be the special configuration of the hyperplanes in $\mathbb{P}^n$ defined as the zero locus of the linear form

$$l_{ij} := \begin{cases} x_i - x_j, & (0 \leq i < j \leq n) \\ x_i. & (0 \leq i < j = n+1) \end{cases}$$

We take the abelian cover Y_m^n of $\mathbb{P}^n$ branched along $\mathcal{A}^n$ corresponding to the function field extention

$$k(\mathbb{P}^n) \subset k(\mathbb{P}^n)\big(\sqrt[m]{l_{i,j}/l_{0,1}} : 0 \leq i < j \leq n+1 \big).$$

Then we can construct a certain rational map $p : Y_m^n \dashrightarrow (X_m^1)^{2n}$ by using certain relations between the linear forms l_{ij}. We take its image $\mathcal{Y}_m^n$ in $(X_m^1)^{2n}$, and the image $\varphi(\mathcal{Y}_m^n)$ under the composite with Abel-Jacobi map $\varphi : (X_m^1)^{2n} \to J(X_m^1)^{2n}$. Our main interest is to study homological properties of these algebraic cycles. These cycles appeared in the C. Schoen's work [5] to investigate Albanese exoticity [4] of Y_m^n. In his paper, he asked a naive question: What (if any) relationship does exist between Y_m^n and the construction of algebraic cycles on abelian varieties of Fermat type by Shioda in [8]? Related to Schoen's question, we pose the following question: What exactly is the canonical decomposition of the fundamental class of $\varphi(\mathcal{Y}_m^n)$ in the cohomology of the Jacobian variety $J(X_m^1)^{2n}$?

In this paper, we adress this question by focusing on the character decomposition of the fundamental class $[\mathcal{Y}_m^n]$ of $\mathcal{Y}_m^n$ in the middle-dimensional primitive cohomology of $(X_m^1)^{2n}$.

Our main results determine all non-zero components which appear in the decomposition of its primitive part $[\mathcal{Y}_m^n]_{\mathrm{prim}}$ in the case $n = 2$. As a direct corollary of the main theorem, we give a partial answer to the above question. To state our results precisely, we prepare some notations. By combining the character decomposition of $H^1(X_m^1)$ and Künneth decomposition, we can see that the middle-dimensional primitive cohomology $H^{2n}_{\mathrm{prim}}((X_m^1)^{2n})$ decomposes into 1-dimensional subspaces:

$$H^{2n}_{\mathrm{prim}}((X_m^1)^{2n}) = \bigoplus_{(\alpha^{(1)},\cdots,\alpha^{(2n)})\in(\mathfrak{A}_m^1)^{2n}} V(\alpha^{(1)},\cdots,\alpha^{(2n)}),$$

where $\mathfrak{A}_m^1$ is a certain subset of the character group $\hat{G}_m^1$ defined in Section 2, and $V(\alpha^{(1)},\cdots,\alpha^{(2n)})$ are eigenspaces with respect to the group action by $(G_m^1)^{2n}$.

Under these notations, the following proposition gives a description of characters which might appear as a non-zero component in the character decomposition of the primitive class $[\mathcal{Y}_m^n]_{\mathrm{prim}}$. In relation to the geometry of Fermat varieties, such components turn out to be originated from linear subvarieties in $2n$-dimensional Fermat variety X_m^{2n}.

Proposition 1.1 (see Proposition 4.5)**.** *Let $(\alpha^{(1)},\cdots,\alpha^{(2n)})$ be a character in $(\mathfrak{A}_m^1)^{2n}$, and $[\mathcal{Y}_m^n]_{\mathrm{prim}}(\alpha^{(1)},\cdots,\alpha^{(2n)})$ be the component of $[\mathcal{Y}_m^n]_{\mathrm{prim}}$ contained in $V(\alpha^{(1)},\cdots,\alpha^{(2n)})$. Then the non-zero condition*

$$[\mathcal{Y}_m^n]_{\mathrm{prim}}(\alpha^{(1)},\cdots,\alpha^{(2n)}) \neq 0$$

leads to the following description of $(\alpha^{(1)},\cdots,\alpha^{(2n)})$:

$$(\alpha^{(1)},\cdots,\alpha^{(2n)}) = \tau_1^*\beta \boxtimes \tau_2^*(-\beta) \tag{1.2}$$

for some $\beta \in \mathfrak{A}_m^n(1) \cap \mathfrak{A}_m^n(2)$. Here $\tau_1^*, \tau_2^* : \hat{G}_m^n \to (\hat{G}_m^1)^n$ are certain group homomorphisms, $\mathfrak{A}_m^n(1), \mathfrak{A}_m^n(2)$ are certain subsets of $\hat{G}_m^n$, which are both defined in Section 2, and $\tau_1^*\beta \boxtimes \tau_2^*(-\beta)$ denotes the element of $(\hat{G}_m^1)^{2n}$ obtained by the product of any $\tau_1^*\beta, \tau_2^*(-\beta) \in (\hat{G}_m^1)^n$.

Our main theorem claims that the converse of Proposition 1.1 is also true in the case $n = 2$. It means that we completely determined the components which appears in the character decomposition of $[\mathcal{Y}_m^2]_{\mathrm{prim}}$.

Theorem 1.3 (see Theorem 4.2)**.** *Consider the case $n = 2$. Let $(\alpha^{(1)},\cdots,\alpha^{(4)})$ be a character in $(\mathfrak{A}_m^1)^4$, and $[\mathcal{Y}_m^2]_{\mathrm{prim}}(\alpha^{(1)},\cdots,\alpha^{(4)})$ be the component of $[\mathcal{Y}_m^2]_{\mathrm{prim}}$ contained in $V(\alpha^{(1)},\cdots,\alpha^{(4)})$. Then we have*

$$[\mathcal{Y}_m^2]_{\mathrm{prim}}(\alpha^{(1)},\cdots,\alpha^{(4)}) \neq 0$$

if and only if

$$(\alpha^{(1)},\cdots,\alpha^{(4)}) = \tau_1^*\beta \boxtimes \tau_2^*(-\beta)$$

for some $\beta \in \mathfrak{A}_m^2(1) \cap \mathfrak{A}_m^2(2)$.

As a corollary, we also completely determined the decomposition of $\varphi_*[\mathcal{Y}_m^2]_{\mathrm{prim}}$, which can be seen as a part of $[\varphi(\mathcal{Y}_m^2)]$ by Lefschetz decomposition (see [9], for example), in the cohomology of $J(X_m^1)^4$. For the precise statement, we remark that the l-th cohomology of the Jacobian variety $J(X_m^1)^{2n}$ admits the decomposition into 1-dimensional subspaces as follows:

$$H^l(J(X_m^1)^{2n}) = \bigoplus_{\lambda \in \Lambda_l} W(\lambda) \tag{1.4}$$

where λ runs in the index set

$$\Lambda_l := \left\{ \left(\{\alpha_1^{(1)}, \cdots, \alpha_{k_1}^{(1)}\}, \cdots \{\alpha_1^{(2n)}, \cdots, \alpha_{k_{2n}}^{(2n)}\}\right) \in \mathcal{P}(\mathfrak{A}_m^1)^{2n} : \sum_i k_i = l \right\},$$

whose elements are $2n$-tuples of subsets in $\mathfrak{A}_m^1$. The cohomology class $\varphi_*[\mathcal{Y}_m^n]_{\mathrm{prim}}$ also decomposes into components corresponding to some $\lambda \in \Lambda_{4ng-2n}$ in (1.4) for $l = 4ng-2n$, where g is the genus of the Fermat curve X_m^1. We obtain the following corollary of Theorem 1.3 on this decomposition in the case $n = 2$.

Corollary 1.5 (see Corollary 4.24)**.** *The decomposition of $\varphi_*[\mathcal{Y}_m^2]_{\mathrm{prim}}$ consists of the non-zero components corresponding to λ which is the complement of $\tau_1^*(\beta) \boxtimes \tau_2^*(-\beta)$ for some $\beta \in \mathfrak{A}_m^2(1) \cap \mathfrak{A}_m^2(2)$, i.e. λ is of the form $(\mathfrak{A}_m^1 \setminus \alpha^{(1)}, \cdots, \mathfrak{A}_m^1 \setminus \alpha^{(4)}) \in \Lambda_{8g-4}$ where we set $(\alpha^{(1)}, \cdots, \alpha^{(4)}) = \tau_1^*(\beta) \boxtimes \tau_2^*(-\beta)$.*

Finally, let us explain the outline of the paper. In Section 2, we recall definitions and some properties of Fermat varieties, and setup some notion necessary for the precise description of the main theorem. In Section 3, we prove the 1-dimensionality result for certain components of the cohomology of a resolution of Y_m^2, which is essentially used in the proof of the main theorem. Finally, in Section 4, we give proofs of our main results.

Acknowledgements. The author would like to thank his adviser, Prof. Tomohide Terasoma, for enormous advice and careful reading of the manuscript.

2. Preliminaries

2.1. **Fermat varieties.** Throughout this paper, let ζ_m be a primitive m-th root of unity, $e^{2\pi i/m}$. Let X_m^n be the n-dimensional Fermat variety of degree m defined by

$$X_m^n = \{(x_0 : \cdots : x_{n+1}) \in \mathbb{P}^{n+1} : x_0^m + \cdots + x_{n+1}^m = 0\}.$$

Let G_m^n be a subgroup in $(\mathbb{C}^*)^{n+2}/(\text{diagonal})$ given by

$$G_m^n := \{(\xi_0 : \cdots : \xi_{n+1}) \in (\mathbb{C}^*)^{n+2}/(\text{diagonal}) : \zeta_m^i = 1, (i = 0, \cdots, n+1)\},$$

which is identified with

$$(\mathbb{Z}/m\mathbb{Z})^{n+2}/(\text{diagonal})$$

by the mapping

$$(\mathbb{Z}/m\mathbb{Z})^{n+2}/(\text{diagonal}) \ni (a_0 : \cdots : a_{n+1}) \mapsto (\zeta_m^{a_0} : \cdots : \zeta_m^{a_{n+1}}) \in G_m^n.$$

This group is naturally acting on the projective space $\mathbb{P}^{n+1}$ and also on the Fermat variety X_m^n. The character group $\hat{G_m^n}$ of the group G_m^n is identified with

$$\{(a_0, \cdots, a_{n+1}) \in (\mathbb{Z}/m\mathbb{Z})^{n+2} : a_0 + \cdots + a_{n+1} = 0\},$$

by the mapping

$$(a_0, \cdots, a_{n+1}) \mapsto \left(G_m^n \ni (\xi_0 : \cdots : \xi_{n+1}) \mapsto \xi_0^{a_0} \cdots \xi_{n+1}^{a_{n+1}} \in \mathbb{C}\right).$$

Let $\mathfrak{A}_m^n$ be a subset of $\hat{G_m^n}$ defined by

$$\mathfrak{A}_m^n = \{\alpha = (a_0, \cdots, a_{n+1}) \in \hat{G_m^n} : a_i \neq 0 \text{ for all } i = 0, \cdots, n+1\}.$$

According to the action of the group G_m^n, the primitive part of the middle cohomology $H^n(X_m^n)$ decomposes as follows [7]:

$$H^n_{\text{prim}}(X_m^n) = \bigoplus_{\alpha \in \mathfrak{A}_m^n} H^n(X_m^n)(\alpha), \quad \dim H^n(X_m^n)(\alpha) = 1, \tag{2.1}$$

where $H^n(X_m^n)(\alpha) := \{x \in H^n(X_m^n) : g^*x = \alpha(g)x \text{ for all } g \in G_m^n\}$ is the eigenspace corresponding to a character $\alpha \in \mathfrak{A}_m^n$.

2.2. **Fermat covers.** We introduce a special hyperplane arrangement $\mathcal{A}^n$ and a covering space Y_m^n as follows.

Definition 2.2. *Let $\mathcal{A}^n := \{H_{i,j}\}_{0 \leq i < j \leq n+1}$ be a family of hyperplanes in the n-dimensional projective space $\mathbb{P}^n$, defined by the following linear forms:*

$$l_{i,j} := \begin{cases} x_i - x_j, & (0 \leq i < j \leq n) \\ x_i. & (0 \leq i < j = n+1) \end{cases}$$

Definition 2.3. *Let Y_m^n be the normalization of $\mathbb{P}^n$ in a field extension*

$$k(\mathbb{P}^n) \subset k(\mathbb{P}^n)(\sqrt[m]{l_{i,j}/l_{0,1}} : 0 \leq i < j \leq n+1). \tag{2.4}$$

Let $\pi : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ be a composition of blow-up morphisms such that the inverse image of $\mathcal{A}^n$ consists of normal crossing divisors, which we can construct by [2]. Let $\widetilde{Y_m^n}$ be the normalization of $\widetilde{\mathbb{P}^n}$ in the field extension (2.4), and let $\varpi : \widetilde{Y_m^n} \to Y_m^n$ be the induced morphism.

We set $N = \binom{n+2}{2}$. According to [5], Y_m^n is given as the following complete intersection in the projective space $\mathbb{P}^{N-1}$:

$$\begin{array}{lll} & Y_m^n & = \{(x_{ij})_{0 \leq i < j \leq n+1} \in \mathbb{P}^{N-1} : x_{i,j}^m - x_{i,n+1}^m + x_{j,n+1}^m = 0, \quad (0 \leq i < j \leq n)\} \\ (2.5) & \downarrow f & \\ & \mathbb{P}^n & = \{(z_{ij})_{0 \leq i < j \leq n+1} \in \mathbb{P}^{N-1} : z_{i,j} - z_{i,n+1} + z_{j,n+1} = 0, \quad (0 \leq i < j \leq n)\}, \end{array}$$

where f maps $(x_{ij})_{i<j} \in Y_m^n$ to $(x_{ij}^m)_{i<j} \in \mathbb{P}^n$. Then the group

$$G_m^{N-2} = \{(\xi_{ij})_{0 \leq i < j \leq n+1} \in (\mathbb{C}^*)^N/(\text{diagonal}) : \xi_{ij}^m = 1 \text{ for all } 0 \leq i < j \leq n+1)\}$$

acts on Y_m^n by

$$G_m^{N-2} \times Y_m^n \to Y_m^n : \quad ((\xi_{ij})_{i<j}, (x_{ij})_{i<j}) \mapsto (\xi_{ij} x_{ij})_{i<j}.$$

2.3. **Rational maps to a Fermat variety - General setup.** First of all, we introduce abstract notions to treat some rational maps from a Fermat cover to a Fermat variety systematically.

Definition 2.6. *Consider the following* $\mathbb{Z}$*-modules generated freely by symbols*

$$A := \bigoplus_{0\le i<j\le n+1} \mathbb{Z}\langle i,j\rangle, \qquad B := \bigoplus_{0\le i\le n+1} \mathbb{Z}\langle i\rangle$$

and define the following maps

$$\mathcal{E}: A \to B, \qquad \sum_{i<j} a_{ij}\langle i,j\rangle \mapsto \sum_{i,j} a_{ij}(\langle i\rangle - \langle j\rangle),$$
$$\mathcal{N}: A \to \mathbb{N}, \qquad \sum_{i,j} a_{ij}\langle i,j\rangle \mapsto \sum_{i,j} |a_{ij}|.$$

Definition 2.7. *For each* $E = \sum_{i,j} a_{ij}\langle i,j\rangle \in Ker\ \mathcal{E}$*, we can define a rational map*

$$p_E : Y_m^n \dashrightarrow X_m^{\mathcal{N}(E)-2}, \qquad (x_{ij})_{0\le i<j\le n+1} \mapsto (\cdots : \underbrace{\varepsilon_{ij}x_{ij} : \cdots : \varepsilon_{ij}x_{ij}}_{|a_{ij}|} : \cdots), \tag{2.8}$$

where ε_{ij} *is given by*

$$\varepsilon_{ij} := \begin{cases} \varepsilon & (a_{ij} < 0), \\ 1 & (0 \le a_{ij}). \end{cases}$$

and ε *is a* m*-th root of* -1*.*

In (2.8), $\varepsilon_{ij}x_{ij}$'s are supposed to be put in the lexicographical order in the set of indices $(i,j) \in \mathbb{Z}^2$, which is

$$(i_1, j_1) < (i_2, j_2) \iff i_1 < i_2 \text{ or } (i_1 = i_2 \text{ and } j_1 < j_2)$$

This rational map p_E is equivariant to the group homomorphism

$$\kappa_E : G_m^{N-2} \to G_m^{\mathcal{N}(E)-2}, \qquad (g_{ij}) \mapsto (\cdots : \underbrace{g_{ij} : \cdots : g_{ij}}_{|a_{ij}|} : \cdots)$$

As in the definition of p_E, the entries g_{ij}'s are supposed to be put in the lexicographical order.

An important example in Ker $\mathcal{E}$ is given by

$$\langle i,j,k\rangle := \langle i,j\rangle - \langle i,k\rangle + \langle j,k\rangle \tag{2.9}$$

for $0 \le i < j < k \le n+1$, and we have a rational map to a Fermat curve

$$p_{\langle i,j,k\rangle} : Y_m^n \dashrightarrow X_m^1, \quad (x_{ij})_{i<j} \mapsto (x_{ij} : \varepsilon x_{ik} : x_{jk}) \tag{2.10}$$

Remark that each $p_{\langle i,j,k\rangle}$ extends to a morphism as follows [5]:

$$\widetilde{p}_{\langle i,j,k\rangle} : \widetilde{Y_m^n} \to X_m^1. \tag{2.11}$$

where $\widetilde{Y_m^n}$ is a resolution of Y_m^n.

Definition 2.12. *Let* E *and* E' *be elements in* A*. A pair* (E, E') *is regular if they are contained in Ker* $\mathcal{E}$ *and satisfy the following relation:*

$$\mathcal{N}(E) + \mathcal{N}(E') - 2 = \mathcal{N}(E + E'). \tag{2.13}$$

Let (E, E') be a regular pair, and we write

$$E = \sum_{i<j} a_{ij}\langle i, j\rangle, \qquad E' = \sum_{i<j} b_{ij}\langle i, j\rangle.$$

To simplify the following argument, we assume that

$$|a_{ij}|, |b_{ij}|, |a_{ij} + b_{ij}| \leq 1, \tag{2.14}$$

which is enough in this paper. Then E and E' satisfy, for some pair $\langle i_0, j_0\rangle$,

$$\begin{cases} a_{i_0j_0} = \pm 1, \quad b_{i_0j_0} = \mp 1, \\ a_{ij} = b_{ij} = 0, \text{ or } |a_{ij}| = 1, |b_{ij}| = 0, \text{ or } |a_{ij}| = 0, |b_{ij}| = 1. \end{cases} \tag{2.15}$$

where i, j run over the range $0 \leq i < j \leq n+1$ and $(i, j) \neq (i_0, j_0)$.

Set $k = \mathcal{N}(E) - 2$ and $k' = \mathcal{N}(E') - 2$. Let us consider $X^k_m, X^{k'}_m$ and $X^{k+k'}$ as hypersurfaces in $\mathbb{P}^{k+1}, \mathbb{P}^{k'+1}$ and $\mathbb{P}^{k+k'+1}$ respectively. We write the homogeneous coordinates for them by $(x_I)_{I\in\Lambda}, (y_J)_{J\in\Lambda'}$ and $(z_K)_{K\in\Lambda''}$, where we set the index sets

$$\begin{aligned} \Lambda &:= \{\langle i, j\rangle : a_{ij} \neq 0\}, \quad \Lambda' := \{\langle i, j\rangle : b_{ij} \neq 0\} \text{ and} \\ \Lambda'' &:= (\Lambda \cup \Lambda') \setminus (\Lambda \cap \Lambda') = (\Lambda \cup \Lambda') \setminus \{\langle i_0, j_0\rangle\}. \end{aligned}$$

Definition 2.16. *Under the above notation, we define*

$$q_{(E,E')} : X^k_m \times X^{k'}_m \dashrightarrow X^{k+k'}_m \tag{2.17}$$

by $q_{(E,E')}((x_I)_{I\in\Lambda}, (y_J)_{J\in\Lambda'}) := (z_K)_{K\in\Lambda''}$, *where we set*

$$z_K := \begin{cases} x_K y_{i_0j_0}, & (K \in \Lambda) \\ \varepsilon x_{i_0j_0} y_K. & (K \in \Lambda') \end{cases}$$

Note that $q_{(E,E')}$ is nothing but the rational map appearing in [6] as the part of the inductive structures of Fermat varieties. Moreover, this rational map $q_{(E,E')}$ is equivariant with respect to the group homomorphism

$$\tau_{(E,E')} : G^k_m \times G^{k'}_m \to G^{k+k'}_m \tag{2.18}$$

defined as follows.

Definition 2.19. *Let E and E' as in Definition 2.16. Let us write the homogeneous coordinates for $G^k_m, G^{k'}_m$ and $G^{k+k'}_m$ by $(g_I)_{I\in\Lambda}, (h_J)_{J\in\Lambda'}$ and $(e_K)_{K\in\Lambda''}$. We define $\tau_{(E,E')}$ by $\tau_{(E,E')}((g_I), (h_J)) := (e_K)$, where we set*

$$e_K := \begin{cases} g_K h_{i_0j_0}, & (K \in \Lambda) \\ g_{i_0j_0} h_K. & (K \in \Lambda') \end{cases}$$

Now we have the following two rational maps by (2.8),

$$\begin{aligned} p_E \times p_{E'} :&\quad Y^n_m \dashrightarrow X^k_m \times X^{k'}_m, \\ p_{E+E'} :&\quad Y^n_m \dashrightarrow X^{k+k'}_m. \end{aligned}$$

We can easily check the commutativity of the following diagram.

$$\begin{array}{ccc} Y_m^n & \xrightarrow{p_E \times p_{E'}} & X_m^{\mathcal{N}(E)-2} \times X_m^{\mathcal{N}(E')-2} \\ & \searrow^{p_{E+E'}} & \downarrow q_{(E,E')} \\ & & X_m^{\mathcal{N}(E+E')-2} \end{array} \qquad \begin{array}{ccc} G_m^{N-2} & \xrightarrow{\kappa_E \times \kappa_{E'}} & G_m^{\mathcal{N}(E)-2} \times G_m^{\mathcal{N}(E')-2} \\ & \searrow^{\kappa_{E+E'}} & \downarrow \tau_{(E,E')} \\ & & G_m^{\mathcal{N}(E+E')-2} \end{array}$$

Moreover, we introduce an operator # as follows.

Definition 2.20. *Let E, E', k and k' be the same as in Definition 2.16. Let $\alpha = (\alpha_I)_{I\in\Lambda}$ and $\beta = (\beta_J)_{J\in\Lambda'}$ be a pair of elements of $\hat{G}_m^k$ and $\hat{G}_m^{k'}$. If α and β satisfy*

$$\alpha_{i_0j_0} = -\beta i_0 j_0, \tag{2.21}$$

where $\langle i_0, j_0\rangle$ is the same as in (2.15). then we define

$$\alpha\#\beta := (\gamma_K)_{K\in\Lambda''},$$

where we set

$$\gamma_K := \begin{cases} \alpha_K, & (K \in \Lambda) \\ \beta_K. & (K \in \Lambda') \end{cases}$$

2.4. **Rational maps to a Fermat variety - Central examples.** Throughout this paper, according to [5], we consider the following two specific rational maps.

Definition 2.22. *Let $p_1, p_2 : Y_m^n \dashrightarrow (X_m^1)^n$ be rational maps defined by*

$$\begin{aligned} p_1 &:= p_{\langle 012\rangle} \times (p_{\langle 013\rangle} \times p_{\langle 024\rangle} \times \cdots \times p_{\langle 0,n-1,n+1\rangle}), \\ &= p_{E_1} \times p_{E_2} \times p_{E_3} \times \cdots \times p_{E_n} \\ p_2 &:= (p_{\langle 123\rangle} \times p_{\langle 234\rangle} \times \cdots \times p_{\langle n-1,n,n+1\rangle}) \times p_{\langle 0,n,n+1\rangle} \\ &= p_{F_1} \times p_{F_2} \times \cdots \times p_{F_{n-1}} \times p_{F_n}, \end{aligned}$$

where we set

$$E_i := \begin{cases} \langle 0,1,2\rangle & (i=1), \\ (-1)^{i-1}\langle 0, i-1, i+1\rangle & (2 \le i \le n), \end{cases}$$

$$F_i := \begin{cases} (-1)^{i-1}\langle i, i+1, i+2\rangle & (1 \le i \le n-1), \\ (-1)^{n-1}\langle 0, n, n+1\rangle & (i = n). \end{cases}$$

Then p_1 and p_2 are dominant and equivariant with the group homomorphisms

$$\kappa_1 := \kappa_{E_1} \times \cdots \times \kappa_{E_n}, \qquad \kappa_2 := \kappa_{F_1} \times \cdots \times \kappa_{F_n},$$

respectively.

We remark that

$$E_1 + \cdots + E_n = F_1 + \cdots + F_n = \langle 1,2\rangle - \langle 1,3\rangle + \cdots + (-1)^{n-1}\langle 0,n\rangle + (-1)^n\langle 0,n+1\rangle$$

in A.

Now we can construct two rational maps q_1 and q_2 which make the following diagram commutative:

$$\begin{array}{ccc} Y_m^n & \overset{p_1}{\dashrightarrow} & (X_m^1)^n \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ (X_m^1)^n & \underset{q_2}{\dashrightarrow} & X_m^n \end{array}$$

In fact, q_1 and q_2 are obtained from rational maps of (2.17) as follows:

$$q_1 := (q_{(E_1+\cdots+E_{n-1},E_n)} \times \pi_{n-2}) \circ \cdots \circ (q_{(E_1+E_2,E_3)} \times \pi_3) \circ (q_{(E_1,E_2)} \times \pi_2),$$
$$q_2 := (q_{(F_1+\cdots+F_{n-1},F_n)} \times \pi_{n-1}) \circ \cdots \circ (q_{(F_1+F_2,F_3)} \times \pi_3) \circ (q_{(F_1,F_2)} \times \pi_2),$$

where $\pi_i : (X_m^1)^{n-i} = X_m^1 \times (X_m^1)^{n-i-1} \to (X_m^1)^{n-i-1}$ is the canonical projection of the last $n-i-1$ factors, and thus q_1 is the iterative composite of rational maps

$$q_{(E_1+\cdots+E_{i-1},E_i)} \times \pi_i : (X_m^{i-1} \times X_m^1) \times (X_m^1)^{n-i} \dashrightarrow X_m^i \times (X_m^1)^{n-i-1}.$$

It is easy to check that

$$q_1 \circ p_1 = q_2 \circ p_2 = p_{E_1+\cdots+E_n} = p_{F_1+\cdots+F_n} =: r,$$

which is given explicitly by

$$r : (x_{ij})_{0\le i<j\le n+1} \mapsto (x_{12} : \varepsilon x_{13} : \cdots : \varepsilon^{n-1} x_{n-1,n+1} : \varepsilon^{n-1} x_{0,n} : \varepsilon^n x_{0,n+1}),$$

where ε is a m-th root of -1. We can make q_1 and q_2 equivariant under group homomorphism

$$\tau_1, \tau_2 : (G_m^1)^n \to G_m^n$$

which are composites of the group homomorphisms of type (2.18).

Set λ be the group homomorphism defined by

$$\lambda := \tau_1 \circ \kappa_1 = \tau_2 \circ \kappa_2 : (g_{ij})_{0\le i<j\le n+1} \mapsto (g_{12} : g_{13} : \cdots : g_{n-1,n+1} : g_{0,n} : g_{0,n+1}).$$

The dual homomorphism of τ_i is denoted by $\tau_i^* : \hat{G_m^n} \to (\hat{G_m^1})^n$ for $i = 1, 2$. We define a subset $\mathfrak{A}_m^n(i)$ of $\hat{G_m^n}$ as follows:

$$\mathfrak{A}_m^n(i) := \mathfrak{A}_m^n \cap (\tau_i^*)^{-1}((\mathfrak{A}_m^1)^n).$$

We can easily chech τ_1 and τ_2 satisfy the following commutative diagram.

$$\begin{array}{ccc} Y_m^n & \overset{p_1}{\dashrightarrow} & (X_m^1)^n \\ {\scriptstyle p_2}\downarrow & \overset{r}{\searrow} & \downarrow{\scriptstyle q_1} \\ (X_m^1)^n & \underset{q_2}{\dashrightarrow} & X_m^n \end{array} \qquad \begin{array}{ccc} G_m^{N-2} & \xrightarrow{\kappa_1} & (G_m^1)^n \\ {\scriptstyle \kappa_2}\downarrow & \overset{\lambda}{\searrow} & \downarrow{\scriptstyle \tau_1} \\ (G_m^1)^n & \xrightarrow[\tau_2]{} & G_m^n \end{array}$$

Here we remark that r is a quotient morphism by

$$G_0 := \mathrm{Ker}\ \lambda \subset G_m^{N-2}. \tag{2.23}$$

3. Cohomology of $\widetilde{Y_m^2}$

Let us consider $Y_m^n, \widetilde{Y_m^n}$ for the case $n = 2$. Then let $N = \binom{n+2}{2} = 6$. According to the action of the group G_m^{N-2} on $H^2(\widetilde{Y_m^2})$, we have the eigenspace decomposition

$$H^2(\widetilde{Y_m^2}) = \bigoplus_{\alpha \in \hat{G}_m^{N-2}} H^2(\widetilde{Y_m^2})(\alpha).$$

Using properties of a quotient morphism $r : Y_m^2 \to X_m^2$ and a Fermat surface X_m^2, we study properties of eigenspaces of $H^2(\widetilde{Y_m^2})$. In the proof of our main theorem in Section 4, the next proposition plays a key role.

Proposition 3.1. *We have*

$$\dim_{\mathbb{C}} H^2(\widetilde{Y_m^2})(\lambda^* \beta) = 1 \tag{3.2}$$

for any $\beta \in \mathfrak{A}_m^2$.

Proof We consider the following diagram

$$\begin{array}{ccccccc} S & \subset & Y_m^2 & \stackrel{r}{\to} & X_m^2 & \supset & \Sigma \\ & & \downarrow f & & \downarrow g & & \\ Z & \subset & \mathbb{P}^2 & \stackrel{\overline{r}}{\to} & \mathbb{P}^2 & \supset & g(\Sigma), \end{array} \tag{3.3}$$

where f and g are quotient morphisms, and Z, Σ, S are defined as follows. Let $Z \subset \mathbb{P}^2$ be a set of triple points in the arrangement $\mathcal{A}^2$ and $\mathring{\mathbb{P}}^2$ be the open subset $\mathbb{P}^2 \setminus Z$. We set

$$\Sigma := g^{-1}(\overline{r}(Z)),$$

and set $\mathring{X}_m^2 := X_m^2 \setminus \Sigma$. Then we have that

$$r^{-1}(\Sigma) = f^{-1}(Z) = \mathrm{Sing}\ Y_m^2 =: S.$$

Let $\mathring{Y}_m^2$ be $Y_m^2 \setminus S$, which is nothing but the smooth part of Y_m^2.

By restricting the diagram (3.3) to the open subsets, we have the following diagram

$$\begin{array}{ccc} \mathring{Y}_m^2 & \stackrel{\mathring{r}}{\to} & \mathring{X}_m^2 \\ \downarrow \mathring{f} & & \downarrow \mathring{g} \\ \mathring{\mathbb{P}}^2 & \to & \mathbb{P}^2 \setminus g(\Sigma), \end{array}$$

where $\mathring{r}, \mathring{f}, \mathring{g}$ are the restricted morphisms. We have $\mathring{r}$ is still a quotient morphism by the subgroup G_0 of the group G_m^{N-2} acting on $\mathring{Y}_m^2$. It follows that the pullback by $\mathring{r}$ gives the embedding

$$\mathring{r}^* : H^*(\mathring{X}_m^2) \xrightarrow{\cong} H^*(\mathring{Y}_m^2)^{G_0} \subset H^*(\mathring{Y}_m^2).$$

Thus we have the following isomorphism between corresponding eigenspaces,

$$H^*(\mathring{X}_m^2)(\beta) \cong H^*(\mathring{Y}_m^2)(\lambda^* \beta), \tag{3.4}$$

for all $\beta \in \mathfrak{A}_m^2(1) \cap \mathfrak{A}_m^2(2)$.

Now we identify the smooth part $\mathring{Y}_m^2$ with an open subset $\varpi^{-1}(\mathring{Y}_m^2)$ in $\widetilde{Y_m^2}$. Then (3.2) follows from the next lemma. □

Lemma 3.5. *We have the following isomorphisms:*

$$H^*(X_m^2)(\beta) \cong H^*(\overset{\circ}{X}{}_m^2)(\beta), \tag{3.6}$$

$$H^2(\widetilde{Y_m^2})(\alpha) \cong H^2(\overset{\circ}{Y}{}_m^2)(\alpha) \tag{3.7}$$

for all $\beta \in \mathfrak{A}_m^2, \alpha \in \mathfrak{A}_m^{N-2}$*. As a consequence, we have*

$$\begin{aligned} \dim H^2(\widetilde{Y_m^2})(\lambda^*\beta) &\overset{(3.7)}{=} \dim H^2(\overset{\circ}{Y}{}_m^2)(\lambda^*\beta) \\ &\overset{(3.4)}{=} \dim H^2(\overset{\circ}{X}{}_m^2)(\beta) \\ &\overset{(3.6)}{=} \dim H^2(X_m^2)(\beta) \\ &\overset{(2.1)}{=} 1. \end{aligned}$$

Proof of (3.6): We consider the long exact sequence of relative cohomology associated to the pair $(X_m^2, \overset{\circ}{X}{}_m^2)$ as follows:

$$\cdots \to H^*(X_m^2, \overset{\circ}{X}{}_m^2)(\beta) \to H^*(X_m^2)(\beta) \to H^*(\overset{\circ}{X}{}_m^2)(\beta) \to \cdots .$$

Then it is enough to show that

$$H^*(X_m^2, \overset{\circ}{X}{}_m^2)(\beta) = 0 \tag{3.8}$$

for any $\beta \in \mathfrak{A}_m^2$. To show this, we use the following fact.

Proposition 3.9 (Alexander duality, see [3] for example)**.** *Let M be a closed orientable manifold of dimension n and K be a locally contractible, compact subspace. Then we have an isomorphism $H_i(M, M \setminus K) \cong H^{n-i}(K)$ for all i.*

Let $\Sigma = X_m^2 \setminus \overset{\circ}{X}{}_m^2$. By Alexander duality, we have an isomorphism

$$H^*(X_m^2, \overset{\circ}{X}{}_m^2)(\beta) \cong H_{4-*}(\Sigma)(\beta)$$

for any $\beta \in \hat{G_m^2}$. It follows that (3.8) is reduced to show that

$$H_*(\Sigma)(\beta) = 0 \tag{3.10}$$

for any $\beta \in \mathfrak{A}_m^2$. Now let $\Sigma_p := g^{-1}(p)$, a fibre of $g : X_m^2 \to \mathbb{P}^2$ over $p \in g(\Sigma)$. We have a decomposition

$$\Sigma = \bigsqcup_{p \in g(\Sigma)} g^{-1}(p) = \bigsqcup_{p \in g(\Sigma)} \Sigma_p.$$

According to this decomposition, the homology $H_*(\Sigma)$ decomposes as

$$H_*(\Sigma) = \bigoplus_{p \in g(\Sigma)} H_*(\Sigma_p). \tag{3.11}$$

Since each Σ_p is a fibre over p, a branch point of g, we have a non-trivial subgroup, for some k,

$$G_p := \{(0 : \cdots : \overset{k\text{-th}}{x} : \cdots : 0) \in G_m^2 : x \in \mathbb{Z}/m\mathbb{Z}\} \subset G_m^2$$

fixing any points in Σ_p. Then G_p acts trivially on $H_*(\Sigma_p)$ and thus we have

$$H_*(\Sigma_p)(\beta) = 0 \tag{3.12}$$

for all $\beta \in \mathfrak{A}_m^{N-2}$. Then we obtain (3.10) and finally the isomorphism (3.6). □

Proof of (3.7): As well as the above argument, we consider the long exact sequence of relative cohomology associated to the pair $(\widetilde{Y_m^2}, \mathring{Y}_m^2)$ as follows:

$$\cdots \to H^2(\widetilde{Y_m^2}, \mathring{Y}_m^2)(\alpha) \to H^2(\widetilde{Y_m^2})(\alpha) \to H^2(\mathring{Y}_m^2)(\alpha) \to \cdots .$$

Since $\dim H^2(\mathring{Y}_m^2)(\alpha) = 1$ by (3.4) and (3.6), the isomorphism (3.7) is reduced to show that

$$H^2(\widetilde{Y_m^2}, \mathring{Y}_m^2)(\alpha) = 0 \tag{3.13}$$

for all $\alpha \in \mathfrak{A}_m^{N-2}$. By Alexander duality, we have an isomorphism

$$H^2(\widetilde{Y_m^2}, \mathring{Y}_m^2)(\alpha) \cong H_2(E')(\alpha), \tag{3.14}$$

where $E' = \widetilde{f}^{-1}(E)$, $E := \pi^{-1}(Z)$ as in the following commutative diagram:

$$\begin{array}{ccccccl} E' & \subset & \widetilde{Y_m^2} & \overset{\varpi}{\to} & Y_m^2 & \supset & S \\ & & \downarrow \widetilde{f} & & \downarrow f & & \\ E & \subset & \widetilde{\mathbb{P}^2} & \overset{\pi}{\to} & \mathbb{P}^2 & \supset & Z \quad = \overline{r}^{-1}(g(\Sigma)) = \{p_1, \cdots, p_4\}, \end{array} \tag{3.15}$$

where $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ is an blowing-up along Z, $\widetilde{f} : \widetilde{Y_m^2} \to \widetilde{\mathbb{P}^2}$ is an abelian covering branched along the total transform $\widetilde{\mathcal{A}^2}$ of the arrangement $\mathcal{A}^2$. Here we remark that the induced morphism $\varpi : \widetilde{Y_m^2} \to Y_m^2$ gives a resolution of Y_m^2 since $\widetilde{A}^2$ consists of normal crossing divisors.

By (3.14), the vanishing (3.13) is reduced to show that

$$H_2(E')(\alpha) = 0 \tag{3.16}$$

for all $\alpha \in \mathfrak{A}_m^{N-2}$. Let $E_z = \pi^{-1}(z)$, a fibre of $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ over $z \in Z$. We have a decomposition

$$E = \bigsqcup_{z \in Z} \pi^{-1}(z) = \bigsqcup_{z \in Z} E_z.$$

Also for $z \in Z$, let $E'_z = \widetilde{f}^{-1}(E_z)$, an inverse image of E_z by $\widetilde{f} : \widetilde{Y_m^2} \to \widetilde{\mathbb{P}^2}$. We have a decomposition

$$E' = \widetilde{f}^{-1}(\pi^{-1}(Z)) = \bigsqcup_{z \in Z} \widetilde{f}^{-1}(E_z) = \bigsqcup_{z \in Z} E'_z$$

Let $B_z = E_z \cap \widetilde{\mathcal{A}^2}$ and let $B'_z = \widetilde{f}^{-1}(B_z) \subset E'_z$. Consider the long exact sequence of the relative homology associated to the pair $(E'_z, E'_z \setminus B'_z)$ as follows:

$$\cdots \to H_2(E'_z, E'_z \setminus B'_z)(\alpha) \to H_2(E'_z)(\alpha) \to H_2(E'_z \setminus B'_z)(\alpha) = 0$$

Here $H_2(E'_z \setminus B'_z)(\alpha) = 0$ follows from the fact that $E'_z \setminus B'_z$ is a 1-dimensional affine curve and Serre's vanishing theorem about the Betti cohomology of smooth affine varieties. Since we have

$$H_2(E'_z, E'_z \setminus B'_z)(\alpha) \cong H^0(B'_z)(\alpha) \tag{3.17}$$

by Alexander duality, the vanishing (3.16) is reduced to show that

$$H^*(B'_z)(\alpha) = 0 \tag{3.18}$$

for all $\alpha \in \mathfrak{A}_m^{N-2}$.

Now we shall show (3.18) as in the argument about (3.16). Let $B'_{z,p} = \widetilde{f}^{-1}(p)$, a fibre of $\widetilde{f}$ over $p \in B_z$. We have a decomposition

$$B'_z = \bigsqcup_{p \in B_z} \widetilde{f}^{-1}(p) = \bigsqcup_{p \in B_z} B'_{z,p}.$$

According to this decomposition, the cohomology $H^*(B'_z)$ decomposes as

$$H^*(B'_z) = \bigoplus_{p \in B_z} H^*(B'_{z,p}).$$

Since each $B'_{z,p}$ is a fibre over p, a branch point of $\widetilde{f}$, we have a non-trivial subgroup, for some k,

$$G'_p := \{(0 : \cdots : 0 : \overset{k\text{-th}}{x} : 0 : \cdots : 0) \in G_m^{N-2} : x \in \mathbb{Z}/m\mathbb{Z}\} \subset G_m^{N-2}$$

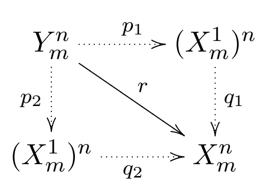
fixing any points in $B'_{z,p}$. Then G'_p acts trivially on $H^*(B'_{z,p})$ and thus we have

$$H^*(B'_{z,p})(\alpha) = 0$$

for all $\alpha \in \mathfrak{A}_m^{N-2}$. Then we obtain (3.18) and finally (3.7). □

4. Character decomposition of $[\mathcal{Y}_m^n]_{\mathrm{prim}}$

4.1. **Statement and proof of Main Theorem.** Let us recall that $\mathcal{Y}_m^n$ is the image under $p = p_1 \times p_2 : Y_m^n \dashrightarrow (X_m^1)^{2n}$, where $p_1 := p_{E_1} \times \cdots \times p_{E_n}$ and $p_2 := p_{F_1} \times \cdots \times p_{F_n}$. Remark that we also have rational maps $q_i : (X_m^1)^n \dashrightarrow X_m^n$ for $i = 1, 2$, which satisfy the following commutative diagram:



To consider the character decomposition of the fundamental class $[\mathcal{Y}_m^n]$, we make some remarks on the cohomology of the product of Fermat curves X_m^1. We recall that the cohomology of X_m^1 admits the following character decomposition with respect to the action of G_m^1:

$$H^*(X_m^1) = \bigoplus_{\alpha \in \hat{G}_m^1} H^*(X_m^1)(\alpha).$$

By combining this decomposition and the Künneth decomposition, we have the following decomposition of the cohomology group $H^{2n}((X_m^1)^{2n})$:

$$\begin{aligned} H^{2n}((X_m^1)^{2n}) &\cong \bigoplus_{s_1+\cdots+s_{2n}=2n} H^{s_1}(X_m^1)\otimes\cdots\otimes H^{s_{2n}}(X_m^1) \\ &= \bigoplus_{s_1+\cdots+s_{2n}=2n} \left(\bigoplus_{\alpha\in\hat{G}_m^1} H^{s_1}(X_m^1)(\alpha)\right)\otimes\cdots\otimes\left(\bigoplus_{\alpha\in\hat{G}_m^1} H^{s_{2n}}(X_m^1)(\alpha)\right) \\ &= \bigoplus_{\substack{s_1+\cdots+s_{2n}=2n,\\ (\alpha^{(1)},\cdots,\alpha^{(2n)})\in(\hat{G}_m^1)^{2n}}} H^{s_1}(X_m^1)(\alpha^{(1)})\otimes\cdots\otimes H^{s_{2n}}(X_m^1)(\alpha^{(2n)}) \end{aligned}$$

We focus on the primitive part $H^{2n}_{\mathrm{prim}}((X_m^1)^{2n})$ whose elements are annihilated by the cup product with its Kähler form, and it follows that

$$H^{2n}_{\mathrm{prim}}((X_m^1)^{2n}) \cong \bigoplus_{(\alpha^{(1)},\cdots,\alpha^{(2n)})\in(\mathfrak{A}_m^1)^{2n}} V(\alpha^{(1)},\cdots,\alpha^{(2n)}), \tag{4.1}$$

where we write $V(\alpha^{(1)},\cdots,\alpha^{(2n)}) := H^1_{\mathrm{prim}}(X_m^1)(\alpha^{(1)})\otimes\cdots\otimes H^1_{\mathrm{prim}}(X_m^1)(\alpha^{(2n)})$, which is a 1-dimensional eigenspace with respect to the group action by $(G_m^1)^{2n}$.

According to (4.1), we have the decomposition of the primitive class $[\mathcal{Y}_m^n]_{\mathrm{prim}}$

$$[\mathcal{Y}_m^n]_{\mathrm{prim}} = \bigoplus_{(\alpha^{(1)},\cdots,\alpha^{(2n)})\in(\mathfrak{A}_m^1)^{2n}} [\mathcal{Y}_m^n]_{\mathrm{prim}}(\alpha^{(1)},\cdots,\alpha^{(2n)}).$$

The main theorem is stated as follows:

Theorem 4.2. *Consider the case $n=2$. Under the above notations,*

$$[\mathcal{Y}_m^2]_{\mathrm{prim}}(\alpha^{(1)},\cdots,\alpha^{(4)}) \neq 0 \tag{4.3}$$

if and only if

$$(\alpha^{(1)},\cdots,\alpha^{(4)}) = \tau_1^*\beta\boxtimes\tau_2^*(-\beta) \tag{4.4}$$

for some $\beta\in\mathfrak{A}_m^2(1)\cap\mathfrak{A}_m^2(2)$. Here, for any $\gamma_1,\gamma_2\in(\hat{G}_m^1)^n$, $\gamma_1\boxtimes\gamma_2$ denotes the element of $(\hat{G}_m^1)^{2n}$ obtained by their product.

Let us start to give proof of Theorem 4.2. First, we show the easier direction (4.3) $\Rightarrow$ (4.4). Moreover, we can prove the n-dimensional version as follows:

Proposition 4.5. *Let $(\alpha^{(1)},\cdots,\alpha^{(2n)})$ be a character in $(\mathfrak{A}_m^1)^{2n}$. If the non-zero condition*

$$[\mathcal{Y}_m^n]_{\mathrm{prim}}(\alpha^{(1)},\cdots,\alpha^{(2n)}) \neq 0. \tag{4.6}$$

holds , then we have

$$(\alpha^{(1)},\cdots,\alpha^{(2n)}) = \tau_1^*\beta\boxtimes\tau_2^*(-\beta) \tag{4.7}$$

for some $\beta\in\mathfrak{A}_m^n(1)\cap\mathfrak{A}_m^n(2)$.

Before giving proof of Proposition 4.5, we notice that the condition (4.6) is equivalent to the following condition

$$\widetilde{p}^*(H^{2n}((X^1_m)^{2n})(-\alpha^{(1)},\cdots,-\alpha^{(2n)}) \neq 0. \tag{4.6 '}$$

Indeed, the equivalence between (4.6) and (4.6 ') follows from the following lemma and its corollary.

Lemma 4.8. *Let M, N be (real) oriented, compact, connected manifolds of dimension $d, 2d$ respectively, $f : M \to N$ be a smooth map and $1_M \in H^0(M;\mathbb{Z})$ a generator. Suppose that N has a group action by an finite abelian group G. Let $f_*(1_M) = \sum_{\alpha\in\hat{G}} v_\alpha$ be a decomposition of the image of 1_M under the Gysin homomorphism $f_* : H^0(M) \to H^d(N)$, according to the eigenspace decomposition*

$$H^d(N;\mathbb{C}) = \bigoplus_{\alpha\in\hat{G}} V(\alpha).$$

Then $v_\alpha \neq 0$ if and only if $f^*(V(-\alpha)) \neq 0$.

Proof Assume that $f^*(V(-\alpha)) \neq 0$. Then since $f_* : H^d(M) \to H^{2d}(N)$ is an isomorphism between the top cohomology, we have

$$f_*f^*(V(-\alpha)) \neq 0$$

Applying the projection formula, we have

$$f_*f^*(V(-\alpha)) = f_*(f^*V(-\alpha) \cup 1_M) = V(-\alpha) \cup f_*(1_M) = V(-\alpha) \cup v_\alpha \neq 0,$$

in which we use the fact $V(\beta) \cup v_\alpha \subset V(\beta) \cup V(\alpha) \subset H^{2d}(N)(\beta+\alpha) = 0$ for all $\beta \in \hat{G}$ satisfying $\beta + \alpha \neq 0$. Thus v_α should be non-zero.

The converse follows by inverting the above argument with Poincaré duality. □

By directly applying this lemma to the condition (4.6), we have the following equivalent condition.

Corollary 4.9. *The condition (4.6) is equivalent to (4.6 ')*

Thus we can work on the condition (4.6 ') instead of (4.6) in proof of both Theorem 4.2 and Proposition 4.5. Now we give proof of $(4.6') \Rightarrow (4.7)$, i.e. Proposition 4.5.

Proof of Proposition 4.5: Throughout in the proof, it is useful to write

$$\alpha^{E_i} := \alpha^{(i)}, \quad \alpha^{F_i} := \alpha^{(n+i)},$$

for $1 \leq i \leq n$. First of all, we note that the G_m^{N-2} acts trivially on the top cohomology $H^{2n}(\widetilde{Y^n_m})$ and thus on $\widetilde{p}^* H^{2n}((X^1_m)^{2n})(\alpha^{E_1},\cdots,\alpha^{F_n})$. Then we have

$$\widetilde{p}^*v = g^*\widetilde{p}^*v = \widetilde{p}^*\kappa(g)^*v = \zeta_m^{\alpha^{E_1}(\kappa_{E_1}(g))+\cdots+\alpha^{F_n}(\kappa_{F_n}(g))}\widetilde{p}^*v, \tag{4.10}$$

for all $g \in G_m^{N-2}$ and $v \in H^{2n}((X^1_m)^{2n})(\alpha^{E_1},\cdots,\alpha^{F_n})$. Since $\widetilde{p}^*v \neq 0$ for some (and any) non-zero vector v by assumption, it follows that

$$\alpha^{E_1}(\kappa_{E_1}(g)) + \cdots + \alpha^{E_n}(\kappa_{E_n}(g)) + \alpha^{F_1}(\kappa_{F_1}(g)) + \cdots + \alpha^{F_n}(\kappa_{F_n}(g)) \equiv 0, \tag{4.11}$$

for any $g \in G_m^{N-2}$, which is the exponent of ζ_m in the right-hand side of (4.10). If we consider (4.11) for generators

$$g = g_{\langle k,l\rangle} := (1 : \cdots : 1 : \zeta_m : 1 : \cdots : 1) \in G_m^{N-2}$$

where ζ_m put in the index of $\langle k,l\rangle$, then we obtain a relation modulo m between integer entries in the α^{E_i} or α^{F_i} corresponding to E_i or F_i which contains $\langle k,l\rangle$ as their non-zero term.

Recall that τ_1^* is obtained as

$$\tau_1^* = (\tau_{(E_1,E_2)} \times \pi_2)^* \circ \cdots \circ (\tau_{(E_1+\cdots+E_{n-1},E_n)} \times \pi_n)^*,$$

where $(\tau_{(E_1+\cdots+E_{i-1},E_i)} \times \pi_i)^* : \hat{G}_m^i \times (\hat{G}_m^1)^{n-i} \to \hat{G}_m^{i-1} \times (\hat{G}_m^1)^{n-i+1}$. To show the Proposition, we will define

$$\alpha^{E_1+\cdots+E_i} \in \hat{G}_m^{\mathcal{N}(E_1+\cdots+E_i)-2}$$

inductively for $i = 1, \cdots, n$, with the relation

$$(\alpha^{E_1+\cdots+E_{i-1}}, \alpha^{E_i}) = \tau^*_{(E_1+\cdots+E_{i-1},E_i)}(\alpha^{E_1+\cdots+E_i}). \tag{4.12}$$

Then it follows that

$$(\alpha^{E_1}, \cdots, \alpha^{E_n}) = \tau_1^*(\alpha^{E_1+\cdots+E_n}). \tag{4.13}$$

since τ_1^* is obtained by the inductive composition of (4.12).

Now we define $\alpha^{E_1+\cdots+E_i}$ inductively as follows. First of all, we remark that the relation (4.12) is empty for $i = 1$. Then suppose that we have $\alpha^{E_1+\cdots+E_{i-1}}$ satisfying (4.12). We can see that $\pm\langle 0, i-1\rangle$ is the common term in $E_1 + \cdots + E_{i-1}$ and E_i. On the other hand, we notice that $F_1, \cdots, F_n$ does not contain the term $\pm\langle 0, i-1\rangle$ for $i = 2, \cdots, n$. Thus, from the relation (4.11) for $g = g_{\langle 0,i-1\rangle}$, it follows that $E_1 + \cdots + E_{i-1}$ and E_i satisfy the condition (2.21) for $\langle i_0, j_0\rangle = \langle 0, i-1\rangle$. Therefore we can define $\alpha^{E_1+\cdots+E_i}$ as

$$\alpha^{E_1+\cdots+E_i} := \alpha^{E_1+\cdots+E_{i-1}} \# \alpha^{E_i}.$$

The relation (4.12) follows from the definition of the operator # and the homomorphism $\tau^*_{(E_1+\cdots+E_{i-1},E_i)}$.

Similarly, we can define $\alpha^{F_1+\cdots+F_i}$ inductively with the relation

$$(\alpha^{F_1}, \cdots, \alpha^{F_n}) = \tau_2^*(\alpha^{F_1+\cdots+F_n}) \tag{4.14}$$

by using the relation (4.11) for $g = g_{\langle 2,3\rangle}, \cdots, g_{\langle n,n+1\rangle}$ as above.

To finish the proof of the Proposition, it is enough to check that

$$\alpha^{E_1+\cdots+E_n} = -\alpha^{F_1+\cdots+F_n}, \tag{4.15}$$

which leads to the desired equation

$$(\alpha^{E_1}, \cdots, \alpha^{E_n}, \alpha^{F_1}, \cdots, \alpha^{F_n}) = \tau_1^*(\beta) \boxtimes \tau_2^*(-\beta)$$

for $\beta := \alpha^{E_1+\cdots+E_n}$ by combining with (4.13) and (4.14). Now (4.15) follows from the relation (4.11) for $g = g_{\langle 1,2\rangle}, \cdots, g_{\langle 0,n+1\rangle}$ corresponding to terms in $E_1 + \cdots + E_n = F_1 + \cdots + F_n = \langle 1,2\rangle - \langle 1,3\rangle + \cdots + (-1)^{n-1}\langle 0,n\rangle + (-1)^n\langle 0,n+1\rangle$. □

We come back to the proof of Theorem 4.2. By Corollary 4.9, we work on the condition

$$\widetilde{p}^*(H^4((X_m^1)^4)(-\alpha^{(1)}, \cdots, -\alpha^{(4)})) \neq 0. \tag{4.3 '}$$

instead of (4.3). In other words, we prove the equivalence between (4.3 ') and (4.4) in the rest of this section. Since Proposition 4.5 contains the direction of $(4.3') \Rightarrow (4.4)$, it remains to show the converse $(4.4) \Rightarrow (4.3')$.

Now we set

$$(\alpha^{(1)}, \cdots, \alpha^{(4)}) = \tau_1^*\beta \boxtimes \tau_2^*(-\beta)$$

for any $\beta \in \mathfrak{A}_m^2(1) \cap \mathfrak{A}_m^2(2)$, and we shall prove (4.3 ') for this $(\alpha^{(1)}, \cdots, \alpha^{(4)})$. The right-hand side of (4.3 ') can be calculated as follows:

$$\begin{aligned} \widetilde{p}^*(H^4((X_m^1)^4)(-\alpha^{(1)}, \cdots, -\alpha^{(4)})) &= \widetilde{p}^*(H^4((X_m^1)^4))(\tau_1^*(-\beta) \boxtimes \tau_2^*\beta) \\ &= \widetilde{p}^*(H^2((X_m^1)^2)(\tau_1^*(-\beta)) \otimes H^2((X_m^1)^2)(\tau_2^*\beta)) \\ &= \widetilde{p_1}^* H^2((X_m^1)^2)(\tau_1^*(-\beta)) \cup \widetilde{p_2}^* H^2((X_m^1)^2)(\tau_2^*\beta) \\ &\subset H^2(\widetilde{Y_m^2})(\lambda^*(-\beta)) \cup H^2(\widetilde{Y_m^2})(\lambda^*\beta) \qquad (4.16) \\ &= H^4(\widetilde{Y_m^2}) \cong \mathbb{C}, \qquad (4.17) \end{aligned}$$

where the last equality (4.17) follows from Poincaré duality.

Claim 4.18. *The inclusion (4.16) is an equality.*

Proof It is enough to show the equality

$$\widetilde{p_i}^* H^2((X_m^1)^2)(\tau_i^*\beta) = H^2(\widetilde{Y_m^2})(\lambda^*\beta), \qquad (4.19)$$

for $i = 1, 2$ and any $\beta \in \mathfrak{A}_m^2(1) \cap \mathfrak{A}_m^2(2)$. Since $\widetilde{p_i}$ is surjective by its definition, it follows that the induced pullback

$$\widetilde{p_i}^* : H^2((X_m^1)^2)(\tau_i^*\beta) \to H^2(\widetilde{Y_m^2})(\lambda^*\beta) \qquad (4.20)$$

is injective. Now since the domain $H^2((X_m^1)^2)(\tau_i^*\beta)$ is not zero and the target $H^2(\widetilde{Y_m^2})(\lambda^*\beta)$ is 1-dimensional by Proposition 3.1, (4.20) gives an isomorphism. Thus (4.19) holds. □

Finally we have

$$\widetilde{p}^*(H^4((X_m^1)^4)(-\alpha^{(1)}, \cdots, -\alpha^{(4)})) = \mathbb{C}$$

and, in particular, (4.3 ') holds. This finishes the proof of Theorem 4.2.

4.2. **Decomposition of** $\varphi_*[\mathcal{Y}_m^n]_{\mathrm{prim}}$. As a corollary of Theorem 4.2, we obtain a result on the cohomology class $\varphi_*[\mathcal{Y}_m^n]_{\mathrm{prim}}$ in the cohomology of the Jacobian variety $J(X_m^1)^{2n}$.

For the precise statement, we recall some facts on the cohomology of the Jacobian variety. First of all, we remark that the first cohomology $H^1(J(X_m^1)^{2n})$ is canonically isomorphic to

$$H^1(J(X_m^1)^{2n}) \cong H^1((X_m^1)^{2n}) \cong \bigoplus_{i=1}^{2n} \pi_i^* H^1(X_m^1) = \bigoplus_{i=1}^{2n} \bigoplus_{\alpha \in \mathfrak{A}_m^1} \pi_i^* V(\alpha), \qquad (4.21)$$

where $\pi_i : (X_m^1)^{2n} \to X_m^1$ is the i-th projection and $V(\alpha)$ is a 1-dimensional eigenspace with respect to a character α. On the other hand, in general, the l-th cohomology of complex torus T is generated by cup products of its first cohomology (see [1] for example).

$$H^l(T) \cong \bigwedge^l H^1(T). \qquad (4.22)$$

Thus we have the canonical decomposition of $H^l(J(X_m^1)^{2n})$ as follows:

$$H^l(J(X_m^1)^{2n}) = \bigoplus_{\lambda \in \Lambda_l} W(\lambda) \tag{4.23}$$

where λ runs in the index set

$$\Lambda_l := \left\{ \left(\{\alpha_1^{(1)}, \cdots, \alpha_{k_1}^{(1)}\}, \cdots \{\alpha_1^{(2n)}, \cdots, \alpha_{k_{2n}}^{(2n)}\}\right) \in \mathcal{P}(\mathfrak{A}_m^1)^{2n} : \sum_i k_i = l \right\},$$

whose elements are $2n$-tuples of subsets in $\mathfrak{A}_m^1$, and we set $W(\lambda)$ a 1-dimensional subspace obtained by the cup product of $\pi_i^* V(\alpha_j^{(i)})$ in (4.21).

For the argument below, we define the complement of $\lambda = (\lambda^1, \cdots, \lambda^{2n}) \in \Lambda_l$ as $\lambda^* := (\mathfrak{A}_m^1 \setminus \lambda^1, \cdots, \mathfrak{A}_m^1 \setminus \lambda^{2n}) \in \Lambda_{4ng-2n}$.

Now we focus on the case $n = 2$. The cohomology class $\varphi_*[\mathcal{Y}_m^2]_{\text{prim}}$ in the cohomology $H^{8g-4}(J(X_m^1)^4)$ also decomposes into components corresponding to $\lambda \in \Lambda_{8g-4}$ in (4.23), where g is the genus of the Fermat curve X_m^1. We obtain the following result on this decomposition as a corollary of Theorem 1.3.

Corollary 4.24. *The decomposition of $\varphi_*[\mathcal{Y}_m^2]_{\text{prim}}$ consists of the non-zero components corresponding to λ which is the complement of $\tau_1^*(\beta) \boxtimes \tau_2^*(-\beta)$ for some $\beta \in \mathfrak{A}_m^2(1) \cap \mathfrak{A}_m^2(2)$, i.e. λ is of the form $(\mathfrak{A}_m^1 \setminus \alpha^{(1)}, \cdots, \mathfrak{A}_m^1 \setminus \alpha^{(4)}) \in \Lambda_{8g-4}$ where we set $(\alpha^{(1)}, \cdots, \alpha^{(4)}) = \tau_1^*(\beta) \boxtimes \tau_2^*(-\beta)$.*

Proof In general, let x be a class in $H^l(J(X_m^1)^{2n})$ and $x(\lambda)$ be a component of the decomposition $x = \sum_\lambda x(\lambda)$ in (4.23). It is easy to see that $x(\lambda) \neq 0$ if and only if $x \cup W(\lambda^*) \neq 0$ holds, because of (4.22) and the definition of $W(\lambda)$ and λ^*.

Let us take a component $\varphi_*[\mathcal{Y}_m^2]_{\text{prim}}(\lambda)$ for $\lambda \in \Lambda_{8g-4}$. We write the complement λ^* as $\lambda^* = (\{\gamma_1^{(1)}, \cdots, \gamma_{k_1}^{(1)}\}, \cdots, \{\gamma_1^{(4)}, \cdots, \gamma_{k_4}^{(4)}\})$, where $k_1 + \cdots + k_4 = 4$ is satisfied. Then it follows that $k_1 = \cdots = k_4 = 1$ and $(\gamma_1^{(1)}, \cdots, \gamma_1^{(4)}) = \tau_1^*\beta \boxtimes \tau_2^*(-\beta)$ for some $\beta \in \mathfrak{A}_m^2(1) \cap \mathfrak{A}_m^2(2)$. Indeed, by the definition of Gysin homomorphism φ_*, the adjoint formula $\left(\varphi_*[\mathcal{Y}_m^2]_{\text{prim}}\right) \cup W(\lambda^*) = [\mathcal{Y}_m^2]_{\text{prim}} \cup \varphi^* W(\lambda^*)$ holds. Since we have $\varphi^* W(\lambda^*) = \pi_1^*\left(V(\gamma_1^{(1)}) \cup \cdots \cup V(\gamma_{k_1}^{(1)})\right) \otimes \cdots \otimes \pi_4^*\left(V(\gamma_1^{(4)}) \cup \cdots \cup V(\gamma_{k_4}^{(4)})\right)$ in $H^4((X_m^1)^4)$, it follows that the non-zero condition $[\mathcal{Y}_m^2]_{\text{prim}} \cup \varphi^* W(\lambda^*) \neq 0$ holds if and only if $k_1 = \cdots = k_4 = 1$ and $\lambda^* = (\gamma_1^{(1)}, \cdots, \gamma_1^{(4)})$ is of the form (4.4) by Theorem 4.2. Thus Corollary 4.24 follows. □

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